Elements of Matrix Algebra

Definition of a Matrix

Definition: **A matrix is a rectangular array of numbers**. These numbers are arranged in rows and columns.

Consider matrix X with R rows and C columns, that is, the **R x C matrix X**. The numbers in this matrix are arranged as follows:

$$\boldsymbol{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1C} \\ x_{21} & x_{22} & \cdots & x_{2C} \\ \cdots & & & \\ x_{R1} & x_{R2} & \cdots & x_{RC} \end{bmatrix} = [x_{ij}] \,, \qquad \textbf{(1)}$$

with i = 1, ..., R and j = 1, ..., C.

Example of a 2 x 3 matrix with its elements:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} . \qquad \textbf{(2)}$$

Types of Matrices

1. Square matrices. **Matrices are square if the number of rows equals the number of columns**. For example, matrices with dimensions 2 x 2 or 5 x 5 are square. The following is an example of a 2 x 2 matrix.

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{12.31} & \mathbf{-4.45} \\ \mathbf{-36.02} & \mathbf{0.71} \end{bmatrix}. \qquad \mathbf{(3)}$$

2. Column vectors. **A matrix with only one column is termed a column vector**. For example, the following are sample dimensions of column vectors: 7 x 1, 3 x 1 and 2 x 1. The following is an example of a 3 x 1 column vector:

$$\boldsymbol{s} = \begin{bmatrix} \boldsymbol{s}_{\mathbf{11}} \\ \boldsymbol{s}_{\mathbf{21}} \\ \boldsymbol{s}_{\mathbf{31}} \end{bmatrix}. \qquad \mathbf{(4)}$$

3. Row vectors. **A matrix with only one row is termed a row vector**. The following are sample dimensions of row vectors: 1 x 2, 1 x 45, 1 x 3. The following is an example of a row vector:

$$\boldsymbol{m} = \begin{bmatrix} \mathbf{114.1} & \mathbf{-32.8} & \mathbf{-1.9} \end{bmatrix} . \qquad \textbf{(5)}$$

4. Diagonal matrices. A square matrix with numbers in its diagonal cells and zeros in its off-diagonal cells is termed **diagonal matrix**. The elements x_{11}, x_{22}, ... describe the **main diagonal** of matrix **X**, that is, the main diagonal of a matrix is constituted by the cells with equal indexes. For example, the diagonal cells of a 3 x 3 matrix have indexes 11, 22, and 33. These are the cells that go from the upper left corner to the lower right corner of a matrix. When a matrix is referred to as diagonal, reference is made to a matrix with a main diagonal. The following is an example of a 3 x 3 diagonal matrix:

$$\boldsymbol{X} = \begin{bmatrix} x_{11} & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{bmatrix} . \qquad \textbf{(6)}$$

Usually, diagonal matrices are written as

$$\boldsymbol{X} = \boldsymbol{diag}\ [x_{11}, x_{22}, ..., x_{nn}]\ , \qquad \textbf{(7)}$$

where n is the number of rows and columns, as diagonal matrices are always square matrices.

5. Scalrar and Identity matrices. A diagonal matrix with elements $d_{11} = d_{22} = d_{33} = ... = d_{nn} = c$ is called **Scalar Matrix**. If, in addition, the constant c = 1, the matrix is termed **Identity Matrix**. The symbol for an n x n identity matrix is I_n, for example

$$\boldsymbol{I_3} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}. \qquad \textbf{(8)}$$

6. Triangular matrices. A square matrix with elements $s_{ij} = 0$ for $j < i$ is termed **upper triangular matrix**. Example of a 2 x 2 upper triangular matrix:

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{u_{11}} & \boldsymbol{u_{12}} \\ \mathbf{0} & \boldsymbol{u_{22}} \end{bmatrix}. \qquad \textbf{(9)}$$

A square matrix with elements $s_{ij} = 0$ for $j > i$ is termed **lower triangular matrix**. Example of a 3 x 3 lower triangular matrix:

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{l_{11}} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{l_{21}} & \boldsymbol{l_{22}} & \mathbf{0} \\ \boldsymbol{l_{31}} & \boldsymbol{l_{32}} & \boldsymbol{l_{33}} \end{bmatrix}. \qquad \textbf{(10)}$$

Diagonal matrices are both upper and lower triangular.

Operations with Matrices I: Transposition

Consider the r x c matrix, **X**. Interchanging rows and columns of **X** yields **X'**, the **transpose** of **X**[1]. By transposing **X**, one moves cell ij to be cell ji. The following example transposes the 2 x 3 matrix, **X**.

$$\boldsymbol{X} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \end{bmatrix} \rightarrow \boldsymbol{X}' = \begin{bmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{5} \\ \mathbf{3} & \mathbf{6} \end{bmatrix}. \qquad \mathbf{(11)}$$

Operations with Matrices II: Addition and Subtraction

Adding and subtracting matrices is only possible if the dimensions of the matrices are the same. For instance, adding **A** with dimensions r_a and c_a and **B** with dimensions r_b and c_b can be performed only if both $r_a = r_b$ and $c_a = c_b$. This is the case in the following example of matrix subtraction.

$$\boldsymbol{L} = \begin{bmatrix} \mathbf{1} & \mathbf{5} \\ \mathbf{10} & \mathbf{3} \end{bmatrix} - \boldsymbol{M} = \begin{bmatrix} \mathbf{4} & \mathbf{3} \\ \mathbf{2} & \mathbf{5} \end{bmatrix} \rightarrow \boldsymbol{L} - \boldsymbol{M} = \begin{bmatrix} \mathbf{1-4} & \mathbf{5-3} \\ \mathbf{10-2} & \mathbf{3-5} \end{bmatrix} = \begin{bmatrix} \mathbf{-3} & \mathbf{2} \\ \mathbf{8} & \mathbf{-2} \end{bmatrix}. \mathbf{(12)}$$

Operations with Matrices III: Multiplication

This module covers three aspects of matrix multiplication:

(1) Multiplication of a matrix with a scalar,

(2) Multiplication of two matrices with each other, and

(3) Multiplication of two vectors.

Multiplication of a matrix with a scalar is performed by multiplying each of its elements with the scalar. For example, multiplication of scalar c = 3 with matrix **L** yields

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[1]Instead of the prime symbol, ', one also finds in the literature the symbol ^{T} to denote a transpose.

$$cL = Lc = 3 * \begin{bmatrix} 1 & 5 \\ 10 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 15 \\ 30 & 9 \end{bmatrix}. \qquad (13)$$

Multiplication of two matrices. Matrices must possess one specific characteristic to be multipliable with each other. Consider the two matrices, **A** and **B**. Researchers wish to multiply them to calculate the matrix product **AB**. This is possible only if the number of columns of **A** is equal to the number of rows of **B**. For example, to be able to **X** with **Y**, **X** must have dimensions z x s and **Y** must have dimensions s x c.

The resulting matrix has the number of rows of X and the number of columns of Y. In the present example, the product **XY** has dimensions z x c.

When multiplying two matrices one follows the following procedure: one multiplies **row by column**, and each element of the row is multiplied by the corresponding element of the column. The resulting products are summed. This sum of products is one of the elements of the resulting matrix with row index carried over from the row of the postmultiplied matrix and column index carried over from the column of the premultiplied matrix.

Consider the following example. A researcher wishes to postmultiply matrix **X** with matrix **Y** where **X** has dimensions 3 x 2 and matrix **Y** has dimensions 2 x 2. Then, the product **XY** of the two matrices can be calculated using the following multiplication procedure:

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}, \; Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \rightarrow XY = \begin{bmatrix} x_{11}*y_{11} + x_{12}*y_{21} & x_{11}*y_{12} + x_{12}*y_{22} \\ x_{21}*y_{11} + x_{22}*y_{21} & x_{21}*y_{12} + x_{22}*y_{22} \\ x_{31}*y_{11} + x_{32}*y_{21} & x_{31}*y_{12} + x_{32}*y_{22} \end{bmatrix}. \qquad (14)$$

Multiplication of two vectors with each other. Everything that was said concerning the multiplication of matrices carries over to the multiplication of two vectors **a** and **b**, with no change. So, no extra rules need to be memorized.

Only the products of a row vector with a column vector and the product of a column vector with a row vector are possible. These two products have special names. The product of a row vector with a column vector, that is **a'b** is called the **inner product**, **dot product**, or **scalar product**.

The product of a column vector with a row vector **ab'** is called the **outer product** (or, simply, **vector product**), and yields a matrix with number of rows equal to the number of elements of **a** and number of columns equal to the number of elements of **b**. In the present context, the inner product is more important.

Consider the following example of the two three-element vectors, **a’** and **b**. **a’** has dimensions 1 x 3 and **b** has dimensions 3 x 1. Multiplication of **a’** with **b** yields the inner product

$$\boldsymbol{a}' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} ;$$

$$\boldsymbol{b} = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} \tag{15}$$

$$\rightarrow \boldsymbol{a}'\boldsymbol{b} = a_{11} * b_{11} + a_{12} * b_{21} + a_{13} * b_{31} = \sum_{i=1}^{3} a_{1i} * b_{i1}.$$

Multiplying **a** with **b**' yields the vector product

$$\boldsymbol{a} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix};$$

$$\boldsymbol{b}' = \begin{bmatrix} b_{11} & b_{12} & b_{13} \end{bmatrix} \tag{16}$$

$$\rightarrow \boldsymbol{ab}' = \begin{bmatrix} a_{11}*b_{11} & a_{11}*b_{12} & a_{11}*b_{13} \\ a_{21}*b_{11} & a_{21}*b_{12} & a_{21}*b_{13} \\ a_{31}*b_{11} & a_{31}*b_{12} & a_{31}*b_{13} \end{bmatrix}.$$

The Rank of a Matrix

A relatively simple method for determining whether rows or columns of a matrix are **linearly independent** involves application of such linear operations as addition/subtraction and multiplication/division. Consider the following example of a 3 x 3 matrix:

$$\boldsymbol{X} = \begin{bmatrix} 1 & -2 & -3 \\ -4 & 2 & 0 \\ 5 & -3 & -1 \end{bmatrix}.$$

The columns of this matrix are linearly dependent. The following operations yield a row vector with only zero elements:

1. Multiply the second column by two; this yields -4, 4, -6;
2. Add the result of Step 1 to the first column; this yields -3, 0, -1;
3. Subtract the third column from the result obtained in the second step; this yields 0, 0, 0.

Thus, the columns are linearly dependent.

If the columns of a matrix are linearly independent, the rows are independent also, and vice versa.

To introduce the concept of **rank of a matrix**, consider matrix **X** with dimensions r x c. The rank of a matrix is defined as the number of linearly independent rows of this matrix. If the rank of a matrix equals the number of columns, that is, rank (**X**) = c, the matrix has **full column rank**. Accordingly, if rank (**X**) = r, the matrix has **full row rank**. If a square matrix has full column rank (and, therefore, full row rank as well), it is said to be **non-singular**. In this case, the inverse of this matrix exists (see the following section).

The Inverse of a Matrix

There is no direct way of performing divisions of matrices. One uses **inverses of matrices** instead. To explain the concept of an inverse consider the two matrices, **A** and **B**. Suppose we postmultiply **A** with **B** and obtain

$$\boldsymbol{AB} = \boldsymbol{I}, \tag{18}$$

where **I** is the identity matrix. Then, we call **B** the inverse of **A**. Usually, inverse matrices are identified by the superscript $^{-1}$.

For an inverse to exist, a matrix must be non-singular, square (although not all square matrices have an inverse), and of full rank.

Calculating an inverse for a matrix can require considerable amounts of computing. Therefore, we do not provide the specific procedural steps for calculating an inverse in general. All major statistical software packages include modules that calculate inverses of matrices. However, we do give examples for two special cases, for which inverses are easily calculated. These examples are the inverses of diagonal matrices and inverses of matrices of 2 x 2 matrices.

The **inverse of a diagonal matrix** is determined by calculating the reciprocal values of its diagonal elements. Consider the 3 x 3 diagonal matrix **A** = diag (a_{11}, a_{22}, a_{33}). The inverse of this matrix is $\mathbf{A}^{-1}$ = diag ($1/a_{11}$, $1/a_{22}$, $1/a_{33}$). This can be illustrated by postmultiplying **A** with $\mathbf{A}^{-1}$ (premultiplying yields the same result):

$$AA^{-1} = \begin{bmatrix} a_{11}*\frac{1}{a_{11}} + 0*0 + 0*0 & a_{11}*0 + 0*\frac{1}{a_{22}} + 0*0 & a_{11}*0 + 0*0 + 0*\frac{1}{a_{33}} \\ 0*\frac{1}{a_{11}} + a_{22}*0 + 0*0 & 0*0 + a_{22}*\frac{1}{a_{22}} + 0*0 & 0*0 + a_{22}*0 + 0*\frac{1}{a_{33}} \\ 0*\frac{1}{a_{11}} + 0*0 + a_{33}*0 & 0*0 + 0*\frac{1}{a_{22}} + a_{33}*0 & 0*0 + 0*0 + a_{33}*\frac{1}{a_{33}} \end{bmatrix} = \quad \textbf{(19)}$$

$$= \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} = \boldsymbol{I_3} \ .$$

Consider the following numerical example of a 2 x 2 diagonal matrix:

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{6} \end{bmatrix} \rightarrow \boldsymbol{A}^{-1} = \begin{bmatrix} \frac{\mathbf{1}}{\mathbf{3}} & \mathbf{0} \\ \mathbf{0} & \frac{\mathbf{1}}{\mathbf{6}} \end{bmatrix} . \quad \textbf{(21)}$$

Multiplying **A** with $\mathbf{A}^{-1}$ results in

$$\boldsymbol{AA}^{-1} = \begin{bmatrix} \mathbf{3}*\frac{\mathbf{1}}{\mathbf{3}} + \mathbf{0}*\mathbf{0} & \mathbf{3}*\mathbf{0}+\mathbf{0}*\frac{\mathbf{1}}{\mathbf{6}} \\ \mathbf{0}*\frac{\mathbf{1}}{\mathbf{3}} + \mathbf{6}*\mathbf{0} & \mathbf{0}*\mathbf{0} + \mathbf{6}*\frac{\mathbf{1}}{\mathbf{6}} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = \boldsymbol{I_2} \ , \quad \textbf{(22)}$$

which illustrates that multiplying a matrix by its inverse yields an identity matrix. The **inverse of**

a 2 x 2 matrix, $\mathbf{X}^{-1}$, can be calculated as follows:

$$\boldsymbol{X}^{-1} = \begin{bmatrix} \dfrac{x_{22}}{x_{11} * x_{22} - x_{12} * x_{21}} & \dfrac{-x_{12}}{x_{11} * x_{22} - x_{12} * x_{21}} \\ \dfrac{-x_{21}}{x_{11} * x_{22} - x_{12} * x_{21}} & \dfrac{x_{11}}{x_{11} * x_{22} - x_{12} * x_{21}} \end{bmatrix}, \quad \textbf{(23)}$$

or, after setting

$$x_{11} * x_{22} - x_{12} * x_{21} = D, \quad \textbf{(24)}$$

we obtain

$$\boldsymbol{X}^{-1} = \begin{bmatrix} \dfrac{x_{22}}{D} & -\dfrac{x_{12}}{D} \\ -\dfrac{x_{21}}{D} & \dfrac{x_{11}}{D} \end{bmatrix}. \quad \textbf{(25)}$$

Inverses of matrices are needed to replace algebraic division. This is most important when solving equations.

Consider the following example. We have the matrix equation **AX** = **BY** that we wish to solve for **Y**. Dividing the entire equation by **B** is not an option because matrix division is not defined. Therefore, we perform the following steps:

1. Premultiply both sides of the equation with $\mathbf{B}^{-1}$. This yields

   $\mathbf{B}^{-1}\mathbf{AX} = \mathbf{B}^{-1}\mathbf{BY}$.

2. Because of $\mathbf{B}^{-1}\mathbf{BY} = \mathbf{IY} = \mathbf{Y}$ we have a solution for the equation. It is

   $\mathbf{Y} = \mathbf{B}^{-1}\mathbf{AX}$.

The Determinant of a Matrix

The **determinant of a matrix** is defined as a function that assigns a real valued number to this matrix. Determinants are defined only for square matrices. Five characteristics of determinants:

1. A matrix **X** is non-singular if and only if the determinant, abbreviated $|\mathbf{X}|$ or det **X**, is different than 0.
2. If a matrix **X** is non-singular, the following holds:

$$\det(A^{-1}) = \frac{1}{\det A}. \tag{26}$$

3. The determinant of a diagonal or triangular matrix equals the product of its diagonal elements:

$$\det(diag(a_{11},\ a_{22},\ \ldots,\ a_{nn})) = a_{11} * a_{22} * \ldots * a_{nn}. \qquad (27)$$

4. The determinant of a product of two matrices equals the product of the two determinants

$$\det(AB) = \det A * \det B.$$

5. The determinant of the transpose of a matrix equals the determinant of the original matrix:

$$\det A' = \det A.$$